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# A Note on Stability of Stochastic Delay Neural Networks*

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**Abstract:** The almost exponential stability for a stochastic recurrent neural network with time-varying delays is discussed by means of a nonnegative semi-martingale convergence theorem, the Lyapunov functional method and the characteristics of stochastic delay recurrent neural networks. The new algebraic criteria of the almost exponential stability for the stochastic recurrent neural network with time-varying delays is derived. These algebraic criteria are simple and practical. Two examples show these new algebraic criteria are better than the relative criteria for the stochastic Hopfield neural network.

**Keywords:** stochastic recurrent neural networks; time-varying delays; almost sure exponential stability; sample Lyapunov exponent

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## 1 Introduction

Recurrent neural networks have many kinds of applications. Wang and Wu[1] presented a multi-layer recurrent neural network for solving continuous-time algebraic matrix Riccati equations in real time. Zhang, Jiang and Wang[2] presented a recurrent neural network for solving the Sylvester equation with time-varying coefficient matrices. Zhang and Ge[3] presented a general recurrent neural network (RNN) model for solving the online inversion of time-varying matrices.

The stability of stochastic neural networks was initiated to study by Liao and Mao in [4,5]. Blythe, Mao and Liao[6] studied the almost sure stability of stochastic Hopfield neural networks. Niu, Zhang and Xu[7] studied exponential stability in mean square for stochastic Cohen-Grossberg neural network with varying delays. Zhang, Xu and Deng[8] studied exponential stability for a stochastic reaction-diffussion neural network with time-varying delays. Jiang, Shen and Liao[9] studied the almost sure stability for a kind of stochastic differential equations with time-varying delays.

This paper discusses almost sure exponential stability of the following stochastic recurrent

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neural networks with time delays

$$\begin{aligned} &dx(t) = \big( - Dx(t) + Af(x(t)) + Bg(x_\tau(t))\big)\mathrm{d}t + \sigma\big(x(t), x_\tau(t), t\big)\mathrm{d}W(t), \\ &x(s) = \xi(s) \quad \text{on} \quad -\overline{\tau} \le s \le 0, \end{aligned} \tag{1}$$

where $D = \mathrm{diag}(d_1, \cdots, d_n)$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $\overline{\tau} = \max\{\tau_i : 1 \le i \le n\}$,

$$f(x) = \big(f_1(x), \cdots, f_n(x)\big)^T, \quad g\big(x_\tau(x)\big) = \big(g_1(x_1(t-\tau_1)), \cdots, g_n(x_n(t-\tau_n))\big)^T,$$

$$\sigma\big(x(t), x_\tau(t), t\big) = \big(\sigma_{ij}(x(t), x_\tau(t), t)\big)_{n\times m}, \quad x_\tau(t) = \big(x_1(t-\tau_1), \cdots, x_n(t-\tau_n)\big)^T,$$

and $\xi(s)$ $(-\overline{\tau} \le s \le 0)$ is a continuous $R^n$-valued stochastic process such that every $\xi(s)$ is $F_0$-measurable and $E|\xi(s)|^2 < \infty$. $n$ denotes the number of neurons in a neural network, $x_i(t)$ corresponds to the state of the $i$th neuron at time $t$, $f_j(x_j(t))$, $g_j(x_j(t))$ denote the activation functions of the $j$th neuron at time $t$, $a_{ij}$ denotes the constant connection weight of the $j$th neuron on the $i$th neuron at time $t$, $b_{ij}$ denotes the constant connection weight of the $j$th neuron on the $i$th neuron at time $t - \tau_j$, $d_i > 0$ represents the rate with which the $i$th neuron will rest its potential to the resting state in isolation when disconnected from the networks and external inputs. $W(t) = (W_1(t), \cdots, W_m(t))^T$ is an $m$-dimensional Brownian motion which is defined on a complete probability space $(\Omega, F, P)$ with a natural filtration $\{F_t\}_{t\ge 0}$ (i.e. $F_t = \sigma\{W(s) : 0 \le s \le t\}$).

Assume $f$, $g$ and $\sigma$ are locally Lipschitz continuous and satisfy the linear growth condition as well. So it is known that (2) has a unique global solution on $t \ge 0$, which is denoted by $x(t;\xi)$. Moreover, assume also that $f(0) = 0$, $g(0) = 0$ and $\sigma(0,0,t) \equiv 0$ for the stability purpose of this note. So (2) admits an equilibrium solution $x(t;0) = 0$.

This paper deals with the almost exponential stability of model (1). The new algebraic criteria of the almost exponential stability for the stochastic recurrent neural network with time-varying delays is derived. These criteria are simple and practical. Furthermore, if $A = 0$ in (1), the model (1) changes into the stochastic Hopfield neural network which was studied in [6]. Two examples in section 3 show that these new algebraic criteria are better than the relative criteria given in [6] for the stochastic Hopfield neural network.

## 2 Main results

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(x,t)$ on $R^n \times R_+$ which are twice continuously differentiable in $x$ and once differentiable in $t$. For each $V \in C^{2,1}(R^n \times R_+; R_+)$, define an operator $\mathcal{L}V$, associated with the stochastic delay neural network (1), from $R^n \times R^n \times R_+$ to $R$ by

$$\mathcal{L}V(x,y,t) = V_t(x,t) + V_x(x,t)\big[ - Dx + Af(x) + Bg(y)\big] + \frac{1}{2}\big[\sigma^T(x,y,t)V_{xx}\sigma(x,y,t)\big],$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \quad V_x(x,t) = \Big(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\Big)^T, \quad V_{xx} = \Big(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Let $C(R^n;R_+)$ denote the family of all continuous functions from $R^n$ to $R_+$, while $C([-\overline{\tau},0];R^n)$ denote the $R^n$-valued continuous functions from $[-\overline{\tau},0]$ to $R^n$.

To obtain our results, we need the following assumptions.

**(H1):** There exist positive constants $\alpha_i$ such that $|f(x_i)|\le\alpha_i|x_i|$, $i=1,2,\cdots,n$.

**(H2):** There exist positive constants $\beta_i$ such that $|g(x_i)|\le\beta_i|x_i|$, $i=1,2,\cdots,n$.

**Theorem 2.1** Assume that there exist a number of functions $V\in C^{2,1}(R^n\times R_+;R_+)$, $\phi_i\in C(R;R_+)$, $\psi_i\in C(R;R_+)\,(1\le i\le n)$ and $3n$ constants $\lambda_i>\mu_i>0$, $\rho_i>0\,(1\le i\le n)$ such that

$$\mathcal{L}V(x,y,t)\le\sum_{i=1}^{n}\big(-\lambda_i\phi_i(x_i)+\mu_i\psi_i(y_i)\big),\quad (x,y,t)\in R^n\times R^n\times R_+, \tag{2}$$

$$V(x,t)\le\sum_{i=1}^{n}\rho_i\phi_i(x_i),\quad (x,t)\in R^n\times R_+, \tag{3}$$

$$\psi_i(x_i)\le\phi_i(x_i),\quad x\in R. \tag{4}$$

Then, for every $\xi\in C([-\overline{\tau},0];R^n)$, the solution of (1) has the property

$$\limsup_{t\to\infty}\frac{1}{t}\log\big(V(x(t;\xi),t)\big)\le-\gamma,\quad a.s.,$$

where $\gamma$ is the root of the equation

$$\gamma=\min_{1\le i\le n}\frac{\lambda_i-\mu_i\mathrm{e}^{\gamma\overline{\tau}}}{\rho_i}.$$

**Proof** Fix initial data $\xi\in C([-\overline{\tau},0];R^n)$ arbitrarily and write simply $x(t;\xi)=x(t)$. Define $U(x,t)=\mathrm{e}^{\gamma t}V(x,t)$ for $(x,t)\in R^n\times R_+$, which is in $C^{2,1}(R^n\times R_+;R_+)$ obviously. We can compute

$$\begin{aligned}\mathcal{L}U(x,y,t) &= \gamma\mathrm{e}^{\gamma t}V(x,t)+\mathrm{e}^{\gamma t}V_t(x,t)+\mathrm{e}^{\gamma t}V_x(x,t)\big[-Dx+Af(x)+Bg(y)\big]\\ &\quad+\frac{1}{2}\mathrm{e}^{\gamma t}\big[\sigma^T(x,y,t)V_{xx}\sigma(x,y,t)\big]\\ &= \mathrm{e}^{\gamma t}\big[\gamma V(x,t)+\mathcal{L}V(x,y,t)\big].\end{aligned}$$

Using conditions (2) and (3), we have

$$\mathcal{L}U(x,y,t)\le\mathrm{e}^{\gamma t}\Big[-\sum_{i=1}^{n}(\lambda_i-\rho_i\gamma)\phi_i(x_i)+\sum_{i=1}^{n}\mu_i\psi_i(y_i)\Big].$$

The Itô formula shows that for any $t\ge0$,

$$\begin{aligned}\mathrm{e}^{\gamma t}V(x,t) &= V\big(x(0),0\big)+\int_0^t\mathcal{L}U\big(x(s),x_\tau(s),s\big)\mathrm{d}s+\int_0^t\mathrm{e}^{\gamma s}V_x\big(x(s),s\big)\sigma\big(x(s),x_\tau(s),s\big)\mathrm{d}W(s)\\ &\le V\big(\xi(0),0\big)-\sum_{i=1}^{n}(\lambda_i-\rho_i\gamma)\int_0^t\mathrm{e}^{\gamma s}\phi_i\big(x_i(s)\big)\mathrm{d}s+\sum_{i=1}^{n}\mu_i\int_0^t\mathrm{e}^{\gamma s}\psi_i\big(x_i(s-\tau_i)\big)\mathrm{d}s\\ &\quad+\int_0^t\mathrm{e}^{\gamma s}V_x\big(x(s),s\big)\sigma\big(x(s),x_\tau(s),s\big)\mathrm{d}W(s).\end{aligned} \tag{5}$$

On the other hand, we have

$$\begin{aligned}\int_{t-\tau_i}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds &= \int_{-\tau_i}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds - \int_0^t e^{\gamma(s-\tau_i)}\psi_i\big(x_i(s-\tau_i)\big)ds \\ &\le \int_{-\bar\tau}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds - \int_0^t e^{\gamma s}\psi_i\big(x_i(s-\tau_i)\big)ds.\end{aligned}$$

This implies

$$\int_0^t e^{\gamma s}\psi_i\big(x_i(s-\tau_i)\big)ds \le \int_{-\bar\tau}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds - e^{\gamma\bar\tau}\int_0^t e^{\gamma s}\psi_i\big(x_i(s)\big)ds. \tag{6}$$

It then follows from (5) and (6) that

$$\begin{aligned}e^{\gamma t}V(x,t) \le\ & V\big(\xi(0),0\big) + \sum_{i=1}^{n}(\lambda_i - \gamma\rho_i + \mu_i e^{\gamma\bar\tau})\int_0^t e^{\gamma s}\phi_i\big(x_i(s)\big)ds \\ & + e^{\gamma\bar\tau}\sum_{i=1}^{n}\mu_i\int_{-\bar\tau}^{0}\psi_i\big(\xi_i(s)\big)ds - e^{\gamma\bar\tau}\sum_{i=1}^{n}\mu_i\int_{t-\tau_i}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds \\ & + \int_0^t e^{\gamma s}V_x\big(x(s),s\big)\sigma\big(x(s),x_\tau(s),s\big)dW(s).\end{aligned}$$

Furthermore, we have that

$$e^{\gamma t}V\big(x(t),t\big) + e^{\gamma\bar\tau}\sum_{i=1}^{n}\mu_i\int_{t-\tau_i}^{t} e^{\gamma s}\psi_i\big(x_i(s)\big)ds \le X(t), \tag{7}$$

where

$$\begin{aligned}X(t) =\ & V\big(\xi(0),0\big) + e^{\gamma\bar\tau}\sum_{i=1}^{n}\mu_i\int_{-\bar\tau}^{0} e^{\gamma s}\psi_i\big(\xi_i(s)\big)ds \\ & + \int_0^t e^{\gamma s}V_x\big(x(s),s\big)\sigma\big(x(s),x_\tau(s),s\big)dW(s),\end{aligned}$$

which is a nonnegative semi-martingale because of $V(x,t)$, $\psi_i(x_i)$, $\mu_i$, $\tau_i$ being nonnegative and inequality (7), and nonnegative semi-martingale convergence theorem[10] shows

$$\lim_{t\to\infty} X(t,\omega) < \infty, \quad a.s..$$

It therefore follows from (7) that

$$\lim_{t\to\infty}\sup\left[e^{\gamma t}V(x,t)\right] < \infty, \quad a.s.,$$

which implies

$$\lim_{t\to\infty}\sup\frac{1}{t}\log\big(V(x(t;\xi),t)\big) \le -\gamma, \quad a.s.$$

as required. The proof is complete.

**Theorem 2.2** Under (H2), assume there exist symmetric nonnegative-definite matrices $C_1$, $C_2$, $C_3 = \mathrm{diag}(\delta_1, \cdots, \delta_n)$ and $C_4$ such that

$$\mathrm{trace}\big[\sigma^T(x,y,t)\sigma(x,y,t)\big] \le x^T C_1 x + g^T(y) C_2 g(y) + y^T C_3 y, \tag{8}$$

$$x^T A f(x) + f^T(x) A^T x \le x^T C_4 x, \tag{9}$$

for all $(x, y, t) \in R^n \times R^n \times R_+$. Assume also that there exists a positive-definite diagonal matrix $G = \mathrm{diag}(g_1, \cdots, g_n)$ such that the symmetric matrix

$$H := \begin{pmatrix} -2D + C_1 + C_3 + C_4 + \bar{D} & B \\ B^T & -G + C_2 \end{pmatrix}$$

is negative-definite, where $\bar{D} = \mathrm{diag}(g_1\beta_1^2, \cdots, g_n\beta_n^2)$. Let $-\lambda = \lambda_{\max}(H)$, the biggest eigenvalue of $H$. So $\lambda > 0$. Then, for every $\xi \in C([-\bar{\tau}, 0]; R^n)$, the sample Lyapunov-exponent of the solution of (1) can be estimated as

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\big(|x(t;\xi)|\big) \le -\frac{\gamma}{2}, \quad a.s., \tag{10}$$

where $\gamma > 0$ is the root of the equation

$$\gamma = \min_{1\le i\le n} \big\{(\lambda + \delta_i + g_i\beta_i^2) - (\delta_i + g_i\beta_i^2 - \lambda\beta_i^2)\mathrm{e}^{\gamma\bar{\tau}}\big\}. \tag{11}$$

**Proof** Let $V(x,t) = |x|^2$. Then the operator $\mathcal{L}V$ has the form

$$\mathcal{L}V = 2x^T\big[-Dx + Af(x) + Bg(y)\big] + \mathrm{trace}\big[\sigma^T(x,y,t)\sigma(x,y,t)\big].$$

By the hypotheses,

$$\begin{aligned}
\mathcal{L}V(x,y,t) &\le -2x^T D x + x^T C_4 x + x^T B g(x) + g^T(x) B^T x + x^T C_1 x + g^T(y) C_2 g(y) + y^T C_3 y \\
&= (x^T\ g^T(y)) H \begin{pmatrix} x \\ f(x) \end{pmatrix} - x^T(C_3 + \bar{D})x + y^T C_3 y + g^T(y) G g(y) \\
&\le -\lambda\big(|x|^2 + |g(y)|^2\big) - x^T(C_3 + \bar{D})x + y^T C_3 y + g^T(y) G g(y) \\
&= -\sum_{i=1}^n (\lambda + \delta_i + g_i\beta_i^2)x_i^2 + \sum_{i=1}^n \big(\delta_i y_i^2 + (g_i - \lambda)g_i^2(y)\big).
\end{aligned}$$

It is easy to see from the construction of $H$ and $-\lambda = \lambda_{\max}(H)$ that $\lambda \le g_i$ for all $1 \le i \le n$. Using (H2) one can then derive that

$$\mathcal{L}V(x,y,t) \le \sum_{i=1}^n \big(-(\lambda + \delta_i + g_i\beta_i^2)x_i^2 + \big(\delta_i y_i^2 + (g_i - \lambda)\beta_i^2 y_i^2\big)\big).$$

In order to apply Theorem 2.1, define $\phi_i$, $\psi_i \in C(R; R_+)$ by

$$\phi_i(x_i) = x_i^2, \quad \psi_i(y_i) = y_i^2, \quad \lambda_i = \lambda + \delta_i + g_i\beta_i^2, \quad \mu_i = \delta_i + (g_i - \lambda)\beta_i^2.$$

It is obvious that

$$V(x,t) = |x|^2 = \sum_{i=1}^{n} \phi_i(x_i) = \sum_{i=1}^{n} \psi_i(x_i), \quad \lambda_i > \mu_i.$$

Moreover

$$\mathcal{L}V(x,y,t) \le \sum_{i=1}^{n} \big( - \lambda_i \phi_i(x_i) + \mu_i \psi_i(y_i) \big).$$

By Theorem 2.1, for every $\xi \in C([-\bar{\tau}, 0], R^n)$, the solution of (1) has the property

$$\limsup_{t\to\infty} \frac{1}{t} \log \big( |x(t;\xi)|^2 \big) \le -\gamma, \quad a.s.,$$

and the required assertion (10) follows. The proof is complete.

In the following, we shall make use of the characteristics of recurrent networks to obtain further results.

**Corollary 2.1** Let (9) and (H2) hold. Assume that there exist nonnegative numbers $\theta_i$ and $\delta_i$ such that

$$\mathrm{trace}\big[\sigma^T(x,y,t)\sigma(x,y,t)\big] \le \sum_{i=1}^{n} [\theta_i x_i^2 + \delta_i y_i^2], \tag{12}$$

for all $(x,y,t) \in R^n \times R^n \times R_+$. Assume also that there exists a positive-definite diagonal matrix $G = \mathrm{diag}(g_1, \cdots, g_n)$ such that the symmetric matrix

$$\bar{H} := \begin{pmatrix} -2D + C_4 + \bar{D} & B \\ B^T & -G \end{pmatrix}$$

is negative-definite, where $\bar{D} = \mathrm{diag}(g_1\beta_1^2, \cdots, g_n\beta_n^2)$. Let $-\bar{\lambda} = \lambda_{\max}(\bar{H})$, the biggest eigenvalue of $\bar{H}$, so $\bar{\lambda} > 0$. If

$$\theta_i + \delta_i < \bar{\lambda}, \quad 1 \le i \le n, \tag{13}$$

then the stochastic delay neural network (1) is almost surely exponentially stable. Moreover, for every $\xi \in C([-\bar{\tau}, 0]; R^n)$, the sample Lyapunov-exponent of the solution of (1) can be estimated as

$$\limsup_{t\to\infty} \frac{1}{t} \log \big( |x(t;\xi)| \big) \le -\frac{\gamma}{2}, \quad a.s., \tag{14}$$

where $\gamma > 0$ is the root of the equation (11), as long as the $\lambda$ in (11) is determined by

$$\lambda = \min_{1\le i\le n} \big[\bar{\lambda} - (\theta_i + \delta_i)\big]. \tag{15}$$

**Proof** Set $C_1 = \mathrm{diag}(\theta_1, \theta_2, \cdots, \theta_n)$, $C_2 = 0$, $C_3 = \mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_n)$. Then (12) can be written as

$$\mathrm{trace}\big[\sigma^T(x,y,t)\sigma(x,y,t)\big] \le x^T C_1 x + g^T(y) C_2 g(y) + y^T C_3 y.$$

In view of Theorem 2.2, it is sufficient to verify that the matrix $H$ defined there is negative-definite. To do so, for any $x,\ y \in R^n$, compute

$$
\begin{aligned}
(x^T\ y^T)H\begin{pmatrix} x \\ y \end{pmatrix} &= (x^T\ y^T)\begin{pmatrix} -2D+C_1+C_3+C_4+\bar{D} & B \\ B^T & -G+C_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= (x^T\ y^T)\bar{H}\begin{pmatrix} x \\ y \end{pmatrix} + (x^T\ y^T)\begin{pmatrix} C_1+C_3 & 0 \\ 0 & C_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\le -\bar{\lambda}(|x|^2+|y|^2) + \sum_{i=1}^{n}(\theta_i+\delta_i)x_i^2 \\
&\le -\lambda(|x|^2+|y|^2),
\end{aligned}
$$

where $\lambda$ is defined by (15) and is positive due to (13). The proof is complete.

**Lemma 2.1** Let $A$ be a real matrix and $P$ be an invertible matrix then

$$x^TAy + y^TA^Tx \le x^TPP^Tx + y^T(P^{-1}A)^T(P^{-1}A)y. \tag{16}$$

**Proof** In fact,

$$
\begin{aligned}
0 &\le |P^Tx - P^{-1}Ay|^2 = (P^Tx - P^{-1}Ay)^T(P^Tx - P^{-1}Ay) \\
&= x^TPP^Tx - x^TAy - y^TA^Tx + y^T(P^{-1}A)^T(P^{-1}A)y.
\end{aligned}
$$

The required assertion (16) follows.

**Corollary 2.2** Let (H1), (H2) and (12) hold. Assume there exist an invertible matrix $P$ and a positive-definite diagonal matrix $G = \mathrm{diag}(g_1, \cdots, g_n)$ such that

$$H_1 := \begin{pmatrix} -2D + PP^T + (P^{-1}AL)^T(P^{-1}AL) + \bar{D} & B \\ B^T & -G \end{pmatrix}$$

is negative-definite, where $\bar{D} = \mathrm{diag}(g_1\beta_1^2, \cdots, g_n\beta_n^2)$, $L = \mathrm{diag}(\alpha_1, \cdots, \alpha_n)$. Let $-\bar{\mu} = \lambda_{\max}(H_1)$, the biggest eigenvalue of $H_1$, so $\bar{\mu} > 0$. If

$$\theta_i + \delta_i < \bar{\mu}, \quad 1 \le i \le n, \tag{17}$$

then the stochastic delay recurrent neural networks (1) is almost surely exponentially stable. Moreover, the sample Lyapunov exponent can be estimated by (14), as long as the $\lambda$ in (11) is determined by

$$\lambda = \min_{1\le i\le n}\left[\bar{\mu} - (\theta_i + \delta_i)\right]. \tag{18}$$

**Proof** Choose $C_4 = PP^T + (P^{-1}AL)^T(P^{-1}AL)$. The conclusion of Corollary 2.2 follows from Corollary 2.1. The proof is complete.

**Corollary 2.6** Let (H1), (H2) and (12) hold. Assume there exist two positive-definite diagonal matrices $Q = \mathrm{diag}(\varepsilon_1, \cdots, \varepsilon_n)$ and $G = \mathrm{diag}(g_1, \cdots, g_n)$ such that

$$H_2 := \begin{pmatrix} -2D + Q + (AL)^T Q^{-1}(AL) + \bar{D} & B \\ B^T & -G \end{pmatrix}$$

is negative-definite, where $\bar{D} = \mathrm{diag}(g_1\beta_1^2, \cdots, g_n\beta_n^2)$, $L = \mathrm{diag}(\alpha_1, \cdots, \alpha_n)$. Let $-\bar{\nu} = \lambda_{\max}(H_2)$, the biggest eigenvalue of $H_2$, so $\bar{\nu} > 0$. If

$$\theta_i + \delta_i < \bar{\nu}, \quad 1 \le i \le n, \tag{19}$$

then the stochastic delay recurrent neural networks (1) is almost surely exponentially stable. Moreover, the sample Lyapunov exponent can be estimated by (14), as long as the $\lambda$ in (11) is determined by

$$\lambda = \min_{1 \le i \le n} \big[\bar{\nu} - (\theta_i + \delta_i)\big]. \tag{20}$$

**Proof** Choose the invertible matrix $P = \mathrm{diag}(\sqrt{\varepsilon_1}, \cdots, \sqrt{\varepsilon_n})$. The conclusion of this Corollary follows from Corollary 2.2. The proof is complete.

## 3 Examples

**Example 3.1** Consider a two-dimensional stochastic delay recurrent neural network

$$\mathrm{d}x(t) = \big[-Dx(t) + Af(x(t)) + Bg(x_\tau(t))\big]\mathrm{d}t + B_1 g\big(x_\tau(t)\big)\mathrm{d}W(t), \tag{21}$$

where $W(t)$ is a real-valued scalar Brownian motion, both $\tau_1$ and $\tau_2$ are positive numbers,

$$A = \begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \sigma(x, y, t) = (0.2y_2, 0.5y_1)^T,$$

while $f_i(x_i) = (\alpha_i x_i \wedge 1) \vee (-1)$ with

$$\alpha_1 = 0.4, \quad \alpha_2 = 0.3, \quad g_i(y_i) = \frac{1 - \mathrm{e}^{-y_i}}{1 + \mathrm{e}^{-y_i}}, \quad i = 1, 2.$$

It is easily shown that (H1) is satisfied with $\alpha_1 = 0.4$, $\alpha_2 = 0.3$ and (H2) is satisfied with $\beta_1 = \beta_2 = 1$, respectively. To apply Theorem 2.2, note in this example such that

$$C_1 = C_2 = 0, \quad C_3 = \mathrm{diag}(0.25, 0.04),$$

$$L = \mathrm{diag}(0.4, 0.3), \quad \sigma^T(x, y, t)\sigma(x, y, t) = 0.25y_1^2 + 0.04y_2^2.$$

Choosing $P = \mathrm{diag}(0.2, 0.2)$ in Lemma 2.1, then $P^{-1} = \mathrm{diag}(5, 5)$, we have

$$x^T Af(x) + f^T(x)Ax \le x^T\big(PP^T + (P^{-1}AL)^T(P^{-1}AL)\big)x,$$

$$C_4 = PP^T + (P^{-1}AL)^T(P^{-1}AL) = \mathrm{diag}(0.12, 0.085), \quad \bar{D} = G,$$

$$-2D + C_1 + C_3 + C_4 = \mathrm{diag}(-7.63, -3.875).$$

Now, choose $D = \mathrm{diag}(3.815, 1.9375)$. The matrix $H$ defined in Theorem 2.2 becomes

$$H = \begin{pmatrix} -3.815 & 0 & 2 & -2 \\ 0 & -1.9375 & 1 & 1 \\ 2 & 1 & -3.815 & 0 \\ -2 & 1 & 0 & -1.9375 \end{pmatrix}.$$

Compute $\lambda_{\max}(H) = -0.1512$ which means that $H$ is negative-definite. By Theorem 2.2, the stochastic delay recurrent neural network (21) is almost exponentially stable. To estimate the sample Lyapunov-exponent, compute, by (11), $\gamma$ satisfied

$$\gamma = 4.2162 - 3.9138\mathrm{e}^{\bar{\tau}\gamma}. \tag{22}$$

If both $\tau_1$ and $\tau_2$ are 0.1 then $\bar{\tau} = 0.1$ and (22) has a unique root $\gamma = 0.2166$. Therefore, Theorem 2.2 shows that the sample Lyapunov-exponent of the solution of network (21) should not be greater than $-0.1083$.

If $A = 0$ in (21), the model (21) changes into the stochastic Hopfield neural network. Choosing $G = \mathrm{diag}(3.875, 1.98)$, we have

$$H = \begin{pmatrix} -3.875 & 0 & 2 & -2 \\ 0 & -1.98 & 1 & 1 \\ 2 & 1 & -3.875 & 0 \\ -2 & 1 & 0 & -1.98 \end{pmatrix}.$$

Compute $\lambda_{\max}(H) = -0.1957$ which means that $H$ is negative-definite. By Theorem 2.2, the the stochastic delay recurrent neural network (21) is almost exponentially stable. To estimate the sample Lyapunov exponent, compute, by (11), $\gamma$ satisfied

$$\gamma = 4.3207 - 3.9293\mathrm{e}^{\bar{\tau}\gamma}. \tag{23}$$

If both $\tau_1$ and $\tau_2$ are 0.1 then $\bar{\tau} = 0.1$ and (23) has a unique root $\gamma = 0.2798$. Therefore, Theorem 2.2 shows that the sample Lyapunov-exponent of the solution of stochastic Hopfield neural network (21) should not be greater than $-0.1399$. Obviously, if $A = 0$ in (21), this upper bound of the sample Lyapunov-exponent is less than the upper bound of the sample Lyapunov-exponent, -0.0834, computed in Example 4.1 in [6].

**Example 3.2** Consider another three-dimensional stochastic delay recurrent neural network

$$\mathrm{d}x(t) = \big[-Dx(t) + Af(x(t)) + Bg(x_\tau(t))\big]\mathrm{d}t + B_1 g(x_\tau(t))\mathrm{d}W(t), \tag{24}$$

where $W(t)$ is a scalar Brownian motion, $B_1$ is a $3\times 3$ constant matrix and

$$D=\mathrm{diag}(3,4,3),\quad A=\begin{pmatrix}-1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 1 & -1\end{pmatrix},\quad B=\begin{pmatrix}-1 & 0 & 2\\ -1 & -2 & 1\\ 1 & -1 & -1\end{pmatrix},$$

$f_i(x_i)=(\alpha_i x_i\wedge 1)\vee(-1)$ with

$$\alpha_1=0.3,\quad \alpha_2=0.4,\quad \alpha_3=0.5,\quad f(x)=\big(f_1(x_1),f_2(x_2),f_3(x_3)\big)^T,$$

$$g(x)=\big(g_1(y_1),g_2(y_2),g_3(y_3)\big)^T,\quad g_i(y_i)=\frac{e^{\beta_i y_1}-e^{-\beta_i y_1}}{e^{\beta_i y_1}+e^{-\beta_i y_1}},\quad i=1,2,3,$$

with $\beta_1=0.4,\ \beta_2=0.5,\ \beta_3=0.4$. Clearly, (H1) and (H2) are satisfied. Note that $\sigma(x,y,t)=B_1 g(y)$ and

$$\sigma^T(x,y,t)\sigma(x,y,t)\le\|B_1\|^2|g(y)|^2\le\|B_1\|^2\big[\beta_1^2y_1^2+\beta_2^2y_2^2+\beta_3^2y_3^2\big]\le 0.25\|B_1\|^2|y|^2.$$

That is, (12) hold with $\theta_i=0,\ \delta_i=0.25\|B_1\|^2$. To apply Corollary 2.3, take $Q=E,\ G=\mathrm{diag}(5,4,5)$, then

$$H_2=\begin{pmatrix}-4.11 & -0.12 & -0.15 & -1 & 0 & 2\\ -0.12 & -5.64 & 0.2 & -1 & -2 & 1\\ -0.15 & 0.2 & -3.45 & 1 & -1 & -1\\ -1 & -1 & 1 & -5 & 0 & 0\\ 0 & -2 & -1 & 0 & -4 & 0\\ 2 & 1 & -1 & 0 & 0 & -5\end{pmatrix}.$$

Compute that $\bar\nu=-\lambda_{\max}(H_2)=1.5294$. Let $\theta_i+\delta_i=0.25\|B_1\|^2<\bar\nu$. Hence, (19) becomes $0.25\|B_1\|^2<1.5294$, namely $\|B_1\|<2.4733$. Corollary 2.3 therefore concludes that the stochastic delay recurrent neural network (24) is almost surely exponentially stable as long as $\|B_1\|<2.4733$.

If $A=0$ in (24), the model (24) changes into the stochastic Hopfield neural network. To apply Corollary 2.3, take $G=\mathrm{diag}(10,8,10)$, then $\bar D=\mathrm{diag}(1.6,2,1.6)$ and

$$H_2=\begin{pmatrix}-2D+\bar D & B\\ B^T & -G\end{pmatrix}=\begin{pmatrix}-4.4 & 0 & 0 & -1 & 0 & 2\\ 0 & -6 & 0 & -1 & -2 & 1\\ 0 & 0 & -4.4 & 1 & -1 & -1\\ -1 & -1 & 1 & -10 & 0 & 0\\ 0 & -2 & -1 & 0 & -8 & 0\\ 2 & 1 & -1 & 0 & 0 & -10\end{pmatrix}.$$

Compute that $\bar\nu=-\lambda_{\max}(H_2)=3.2622$. Let $\theta_i+\delta_i=0.25\|B_1\|^2<\bar\nu$. Hence, (19) becomes $0.25\|B_1\|^2<3.2622$, namely $\|B_1\|<3.6123$. Corollary 2.3 therefore concludes that the stochastic Hopfield neural network (24) is almost surely exponentially stable as long as $\|B_1\|<3.6123$.

Obviously, if $A = 0$ in (24), this upper bound of $\|B_1\|$ is bigger than the upper bound of $\|B_1\|$, 3.189, computed in Example 4.3 in [6].

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# 关于随机时滞神经网络稳定性的注记

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**摘 要**: 利用非负鞅收敛定理、Lyapunov泛函方法和网络自身的特性讨论了变时滞随机递归神经网络的随机指数稳定性，给出了这类神经网络随机指数稳定性的新的代数准则，所给代数准则简单易用。两个应用实例说明即使针对随机Hopfield神经网络所给的代数准则也优于相关的判别准则。

**关键词**: 随机递归神经网络；变时滞；随机指数稳定性；样本Lyapunov指数